Lecture 21

In this lecture we complete our study of propositional logic and begin a brief study of predicate logic.

E.g. Construct the truth table for the compound propositions $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$.

Sol$^{\underline{n}}$

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Q$^{\underline{n}}$ What is the relationship between these two compound prop$^{\underline{n}}$s?

Answer $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

This is one of the <u>de Morgan laws</u> for logic.

The other is:

$$\underbrace{\neg(P \wedge Q) \equiv \neg P \vee \neg Q}.$$

<u>Exercise</u>: show this using truth tables.

<u>Consequence</u>

Any expression using "or" can be replaced by an expression using "and" and "not".

I'll have a hamburger <u>or</u> a slice of pizza for lunch

$\equiv$ It's not true that I won't have a hamburger and that I won't have a slice of pizza for lunch.

Hierarchy of Connectives

$$\neg$$
$$\wedge$$
$$\vee$$
$$\Rightarrow$$
$$\Leftrightarrow$$

$(\neg P) \wedge Q$

$\neg P \wedge Q$ means $(\neg P) \wedge Q$. *etc.*

$A \wedge B \vee C$ means $(A \wedge B) \vee C$

But because $\wedge$ and $\vee$ look so similar, we almost always use brackets in this sort of expression.

<u>E.g.</u> $P \wedge Q \Rightarrow P \vee R$

P	Q	R	$P \wedge Q$	$P \vee R$	$P \wedge Q \Rightarrow P \vee R$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	F	T	T
F	F	F	F	F	T

non-essential columns

<u>Note</u> This is an example of a <u>tautology</u> which means $P \wedge Q \Rightarrow P \vee R$ is always true.

If the last column in a truth table has only F (for false), then the compound propⁿ is called a <u>contradiction</u>.

<u>E.g.</u> Show that $(P \wedge Q) \wedge \sim P$ is a contradiction.

If a compound propⁿ is neither a tautology nor a contradiction then the last column of the truth table will have both T and F appearing.

Such a compound propⁿ is called a <u>contingency</u>.

<u>Note</u>

Several logical equivalences have been established:

$$P \Rightarrow Q \equiv \neg P \vee Q$$

$$P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

$$\left.\begin{aligned} \neg(P \wedge Q) &\equiv \neg P \vee \neg Q \\ \neg(P \vee Q) &\equiv \neg P \wedge \neg Q \end{aligned}\right\} \text{de Morgan's laws}$$

A consequence is that all compounds prop$^{\underline{n}}$s can be expressed using only two connectives: negation & conjunction, or negation & disjunction, or negation & implication.

<u>E.g.</u> <u>If</u> dogs have humps <u>then</u> the moon is green

$\equiv$ Dogs don't have humps or the moon is green.

Predicate Logic

We contrast <u>propositions</u> & <u>predicates</u>.

A proposition has to be unambiguously true or false.

In contrast, a <u>predicate</u> is a statement involving at least one variable x (say). The truth value may depend on the value of x.

<u>E.g.</u> Let $P(x)$ mean "x is an integer".

Then $P(2)$ is true, but $P(\pi)$ is false.

_E.g._

Let $Y(t)$ mean "my friend t has yellow hair".
Then $Y(\text{Emily})$ may be true, while
$Y(\text{Steve})$ may be false.

What values is the variable allowed to take?
There may be several variables. E.g., let
$T(x,y)$ mean x and y are relatively prime.

The variables have to range over some set D, called the <u>domain of interpretation</u> or the <u>universe of discourse</u>.

E.g. Let the domain of interpretation for a predicate $T(x, y)$ be $D = \mathbb{Z}$ = the set of all integers. Let $T(x, y)$ mean that x and y are relatively prime.

Then $T(10, 21)$ is true

while $T(12, 15)$ is false.

Why? $T(10, 21)$ means "10 and 21 are relatively prime".

This is true.

$T(12, 15)$ means "12 and 15 are relatively prime".

This is false, because $3 | 12$ and also $3 | 15$.

It's natural to introduce the idea of a <u>quantifier</u> when we're considering predicates. These tell us how often the predicate is true.

$\forall$ — "for all"

$\exists$ — "there exists"

Predicate logic involves statements like these:

$\forall x\ P(x)$ [for all x, $P(x)$ is true]

$\forall x\ \exists y\ P(x,y)$ [for all x there exists y such that $P(x,y)$ is true]

We study these in detail next time.